3. písemka MAB2

-1-

① $\int_0^{\sqrt{3}} x \cdot \operatorname{arctg}(x)\, dx = \left| \begin{matrix} f(x) = \operatorname{arctg}(x) \to f'(x) = \frac{1}{1+x^2} \\ g'(x) = x \to g(x) = \frac{x^2}{2} \end{matrix} \right| \overset{PP}{=} \left[\frac{x^2}{2} \cdot \operatorname{arctg}(x)\right]_0^{\sqrt{3}} -$

$- \int_0^{\sqrt{3}} \frac{1}{2} \frac{x^2}{1+x^2}\, dx = \frac{1}{2}\left[0 \cdot 0 + \frac{3}{1} \cdot \operatorname{arctg}(\sqrt{3})\right] - \frac{1}{2} \cdot \int_0^{\sqrt{3}} \frac{x^2+1-1}{1+x^2}\, dx =$

$\operatorname{arctg}(\sqrt{3}) = y \Leftrightarrow \operatorname{tg}(y) = \sqrt{3} \overset{!}{=} \frac{\sin(y)}{\cos(y)} \overset{!}{=} \frac{\sqrt{3}}{1} = \frac{\frac{\sqrt{3}}{2}}{1/2} \Leftrightarrow y = \frac{\pi}{3}$

$= \frac{3}{2} \cdot \frac{\pi}{3} - \frac{1}{2} \int_0^{\sqrt{3}} \left(1 - \frac{1}{1+x^2}\right) dx =$

$= \frac{\pi}{2} - \frac{1}{2} \cdot \left[x - \operatorname{arctg}(x)\right]_0^{\sqrt{3}} = \frac{\pi}{2} - \frac{1}{2}\left[\sqrt{3} - \underbrace{\operatorname{arctg}(\sqrt{3})}_{=\frac{\pi}{3}} - 0 + \operatorname{arctg}(0)\right] =$

$= \frac{\pi}{2} - \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} = \frac{2}{3}\pi - \frac{\sqrt{3}}{2}$

② $p > 0$:

$\lim_{n\to+\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \lim_{n\to+\infty} \frac{\sum_{k=1}^{n} k^p}{n^{p+1}} = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n} \cdot \left(\frac{k}{n}\right)^p =$

$= \left| \begin{matrix} \text{ZLIP: } x_k = a + \frac{b-a}{n} \cdot k & \frac{b-a}{n} = \frac{1}{n} \Rightarrow b-a = 1 \Rightarrow \text{volba: } a=0,\ b=1 \\ \int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k) \cdot \frac{b-a}{n} & x_k = 0 + \frac{1-0}{n} \cdot k \\ & x_k = \frac{k}{n} \Rightarrow f(x_k) = x_k^p \end{matrix} \right| =$

$= \int_0^1 x^p\, dx = \left[\frac{x^{p+1}}{p+1}\right]_0^1 = \frac{1-0}{p+1} = \frac{1}{p+1}$

③ $\lim_{x\to+\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} \overset{L'H✓}{=} \lim_{x\to+\infty} \frac{2 \cdot \left(\int_0^x e^{t^2} dt\right) \cdot \left(\int_0^x e^{t^2} dt\right)'}{\left(\int_0^x e^{2t^2} dt\right)'} = \left| \begin{matrix} \text{integrál} \\ \text{jako fce} \\ \text{meze} \end{matrix} \right| =$

integr. ✓

integrab. ✓

$= \lim_{x\to+\infty} \frac{2 \cdot \int_0^x e^{t^2} dt \cdot e^{x^2}}{e^{2x^2}} = 2 \cdot \lim_{x\to+\infty} \frac{\int_0^x e^{t^2} dt}{e^{x^2}} \overset{L'H✓}{=} 2 \cdot \lim_{x\to+\infty} \frac{e^{x^2}}{e^{x^2} \cdot 2x} = 0$

④ $\lim\limits_{\varepsilon\to 0^+}\int_{\varepsilon}^{2\varepsilon}\frac{e^x}{x}\sin x\,dx$ pomocí 1. věty o střední hodnotě

$f(x)=e^x$ ... integr., nezáp. na $\langle\varepsilon;2\varepsilon\rangle$ ✓

$g(x)=\frac{\sin x}{x}$ ... spoj. ⇒ integr. ✓ (na $\langle\varepsilon;2\varepsilon\rangle$)

klesající

$\inf\limits_{\langle\varepsilon;2\varepsilon\rangle} g(x)=\frac{\sin(2\varepsilon)}{2\varepsilon}$; $\sup\limits_{\langle\varepsilon;2\varepsilon\rangle} g(x)=\frac{\sin(\varepsilon)}{\varepsilon}$

$\Rightarrow \exists\,\mu\in\left\langle\frac{\sin(2\varepsilon)}{2\varepsilon};\frac{\sin(\varepsilon)}{\varepsilon}\right\rangle$ tak, že

$$\int_{\varepsilon}^{2\varepsilon} fg=\mu\cdot\int_{\varepsilon}^{2\varepsilon}e^x\,dx=\mu\cdot\left[e^x\right]_{\varepsilon}^{2\varepsilon}=\mu\cdot\left((e^{\varepsilon})^2-e^{\varepsilon}\right)=\underset{\mu(\varepsilon)}{\underset{\shortparallel}{\mu}}\cdot e^{\varepsilon}(e^{\varepsilon}-1)$$

$$\underbrace{\frac{\sin(2\varepsilon)}{2\varepsilon}}_{\downarrow\,\varepsilon\to0^+ \atop 1}\le\mu(\varepsilon)\le\underbrace{\frac{\sin(\varepsilon)}{\varepsilon}}_{\downarrow\,\varepsilon\to0^+ \atop 1}\quad\Rightarrow\quad\lim_{\varepsilon\to0^+}\mu(\varepsilon)=1$$

$$\Rightarrow\lim_{\varepsilon\to0^+}\int_{\varepsilon}^{2\varepsilon}f(x)g(x)\,dx=\lim_{\varepsilon\to0^+}\mu(\varepsilon)\cdot e^{\varepsilon}(e^{\varepsilon}-1)\overset{\text{VOAL}}{=}\lim_{\varepsilon\to0^+}\mu(\varepsilon)\cdot\lim_{\varepsilon\to0^+}e^{\varepsilon}(e^{\varepsilon}-1)=$$

$$=1\cdot e^0\cdot(e^0-1)=1\cdot1\cdot0=\underline{\underline{0}}$$